San Diego Math League Gauss Olympiad

March 29, 2008

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1. (a) Find the arithmetic sequence with first term 1 and common difference not equal to 0, whose 2^{nd}, 10^{th}, and 34^{th} terms are the first three terms in a geometric sequence.

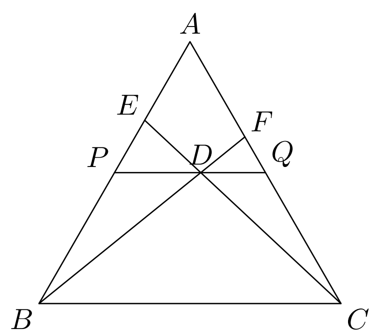
   (b) The fourth term in the geometric sequence in part (a) appears as the n^{th} term in the arithmetic sequence in part (a). Find the value of n.

2. The set S contains five distinct positive integers. If pairs of distinct elements of S are added, then the following ten sums are obtained: 17, 22, 23, 24, 25, 30, 33, 34, 39, and 41. What are the elements in S?

3. Let ABC be an equilateral triangle, and let P and Q be the midpoints of sides $\overline{AB}$ and $\overline{AC}$, respectively. Let D be a point on $\overline{PQ}$. Extend the lines $\overline{CD}$ and $\overline{BD}$ so that they meet $\overline{AB}$ and $\overline{AC}$ at E and F, respectively. Show that the value of

$$\frac{BC}{EB} + \frac{BC}{FC}$$

is independent of the location of point D, and find this value.



4. Find all ordered pairs of integers (x, y) that satisfy the equation $7(x + y) = 3(x^2 - xy + y^2)$.

5. Let I, I_1, I_2, $\ldots$, I_n be closed intervals, such that I contains I_m for all $1 \le m \le n$, and the union of the n intervals I_1, I_2, $\ldots$, I_n is the interval I. Show that the union of the left halves of the n intervals I_1, I_2, $\ldots$, I_n contains at least half of the interval I.

   Note: A closed interval is a set of real numbers of the form $\{x : a \le x \le b\}$, which is denoted by $[a, b]$. The left half of the closed interval $[a, b]$ is the interval $[a, \frac{a+b}{2}]$.